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Hyper β-languages of order n

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Abstract. β-languages of order n have been introduced by the authors of [9]. In [5], the authors have shown that the class of β-languages of order n is closed under union, concatenation and star-closure operations. In this paper, we show that the class of β-languages of order n is not closed under substitution operation. Motivated by this, we introduce the notion of hyper β-languages of order $n(n \geq 1)$ and show that the class of hyper β-languages of order n is closed under substitution, union, concatenation and star-closure operations.

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1. Introduction

The authors of [9] have introduced the concept of β-grammar and β-languages of order n. The class of β-languages of order $n(n \geq 1)$ lies between non-deterministic context-free languages and deterministic context-free languages and therefore contain all regular languages. In [5], the authors have shown that the class of β-languages of order n is closed under union, concatenation and star-closure operations.

In the present paper, we begin with the study of closure properties of β-languages with respect to the substitution operation. In this operation, we replace each symbol in the strings of one language by the chosen entire languages (which may be different also). We show that the class of β-languages of order n is not closed under substitution operation. Motivated

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by this, we then introduce a more generalized class viz. class of hyper β-languages of order n and show that the class of hyper β-languages of order n is closed under substitution, union, concatenation and star-closure operations.

2. Preliminaries

In this section, we present some definitions available in the literature:

Definition 2.1 [12].

(i) A finite non-empty set Σ is called an **"alphabet"**.

(ii) A **"string"** is a finite sequence of symbols from the alphabet.

(iii) The **"concatenation"** of two strings u and v is the string obtained by appending the symbols of u to the right end of v.

(iv) The **"length"** of string w denoted by $|w|$ is the number of symbols in the string.

(v) An **"empty string"** is a string with no symbol in it. It is denoted by λ and $|\lambda| = 0$.

(vi) If Σ is any alphabet, then Σ^k $(k \geq 0)$ denotes the set of all strings of length k with symbols from Σ.

(vii) The set of all strings over an alphabet Σ is denoted by Σ^*, i.e.

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \cdots .$$

(viii) The set of all non-empty strings from the alphabet Σ is denoted by Σ^+ and is given by

$$\Sigma^+ = \Sigma^* - \{\lambda\} = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \cdots$$

(ix) A **"language"** L over an alphabet Σ is defined as a subset of Σ^*.

(x) A string in a language L is called a **"sentence"** of L.

(xi) The **"union"**, **"intersection"** and **"difference"** of two languages are defined in the set theoretic way.

(xii) The **"complement"** of a language L over an alphabet Σ is defined as $\bar{L} = \Sigma^* - L$.

(xiii) The **"concatenation"** of two languages L_1 and L_2 is the set of all strings obtained by concatenating a string of L_1 with a string of L_2, i.e.

$$L_1L_2 = \{uv | u \in L_1 \text{ and } v \in L_2\}.$$

(xiv) The **"star-closure"** of a language L is defined as

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots .$$

Also, the **"positive-closure"** of a language L is given by

$$L^+ = L^1 \cup L^2 \cup \cdots .$$

(xv) A **"grammar"** G is defined as a quadruple

$$G = (V, T, S, P),$$

where V is a finite set of objects called **"variables"**, T is a finite set of objects called **"terminal symbols"** with $V \cap T = \phi$, $S \in V$ is a special symbol called the **"start"** symbol, P is a finite set of **"productions"** of the form $x \to y$ where $x \in (V \cup T)^+$ and $y \in (V \cup T)^*$.

(xvi) We say that the string $w = uxv$ **"derives"** the string $z = uyv$ if the string z is obtained from w by applying the production $x \to y$ to w. This is written as $w \Rightarrow z$. If

$$w_1 \Rightarrow w_2 \Rightarrow \cdots \Rightarrow w_n,$$

then we say that w_1 derives w_n and write $w_1 \Rightarrow^* w_n$.

(xvii) Let $G = (V, T, S, P)$ be a grammar. Then the **"language"** $L(G)$ generated by G is given by

$$L(G) = \{w \in T^* | S \Rightarrow^* w\}.$$

(xviii) If $w \in L(G)$, then the sequence

$$S \Rightarrow w_1 \Rightarrow w_2 \Rightarrow \cdots \Rightarrow w_n \Rightarrow w.$$

is a **"derivation"** of the sentence w. The strings $S, w_1, w_2, \cdots, w_n$ which contain variables as well as terminals are called **"sentential forms"** of the derivation.

(xix) A grammar $G = (V, T, S, P)$ is said to be **"right-linear" (resp. left-liner)** if all productions in G are of the form

$$A \to xB \text{ (resp.} A \to Bx),$$

or

$$A \to x,$$

where $A, B \in V$ and $x \in T^*$. A **"regular grammar"** is one that is either right linear or left linear.

Definition 2.2 [9]. A context-free grammar $G = (V, T, S, P)$ is said to be a **"β-grammar of order n"** $(n \geq 1)$ if all productions in P are of the form $A \to ax$ where $a \in T \cup \{\lambda\}$ and $x \in V^*$ and any pair (A, a) occurs at most "n'' times in P. A β-grammar of order n is denoted by $\beta(n)$.

Definition 2.3 [9]. The language generated by a β-grammar of order n is called a **"β-language of order n"**.

Now we define **"substitution"** operation as follows:

Definition 2.4 [3]. Let Σ be an alphabet and suppose that for every $a \in \Sigma$, there is a chosen language L_a. The languages L_a for each $a \in \Sigma$ can be over any alphabets, not necessarily Σ and not necessarily the same. This choice of languages defines a function f_s **(substitution function)** on Σ, and we shall refer L_a as $f_s(a)$ for each symbol $a \in \Sigma$. If $w = a_1 a_2 \cdots a_n$ is a string in Σ^*, then $f_s(w)$ is the language of all strings of the form $x_1 x_2 \cdots x_n$ such that $x_i \in f_s(a_i)$ for $i = 1, 2, \cdots n$.

In other words, $f_s(w)$ is the concatenation of the languages $f_s(a_1) f_s(a_2) \cdots f_s(a_n)$. We can further extend the definition of substitution function f_s to a language L as follows:

$$f_s(L) = \bigcup_{w \in L} f_s(w).$$

Example 2.5. Le $\Sigma = \{0, 1\}$. Let $f_s(0) = \{ab\}$ and $f_s(1) = \{aa, ba\}$ i.e. the language $f_s(0)$ is the finite language consisting of single string ab while $f_s(1)$ is the finite language consisting of two strings aa and ba.

Let $w = 10$. Then $f_s(w)$ is the concatenation of the langauge $f_s(1) f_s(0)$. That is

$$f_s(w) = f_s(1) f_s(0) = \{aaab, baab\}.$$

3. Non-closure of β-languages of order n under substitution operation

In this section, we show that the class of β-languages of order n is not closed under substitution operation.

Theorem 3.1. *If L is a β-language of order n over an alphabet Σ and f_s is a substitution function on Σ such that $f_s(a)$ is a β-language of order n for each $a \in \Sigma$, then $f_s(L)$ is a β-language of order greater than n, or equivalently, the class of β-language of order n is not closed under substitution operation.*

Proof. Let $G = (V, \Sigma, P, S)$ be a β-grammar of order n for the given β-language L.
For each $a \in \Sigma$, let $G_a(V_a, T_a, P_a, S_a)$ be a β-grammar of order n for the β-language $f_s(a)$.
Without any loss of generality, we assume that

$$V \cap V_a = \phi \text{ and } V_a \cap V_b = \phi \text{ for each } a, b \in \Sigma;$$

$$T \cap T_a = \phi \text{ and } T_a \cap T_b = \phi \text{ for each } a, b \in \Sigma.$$

We construct a new β-grammar $G' = (V', T', P', S)$ for the language $f_s(L)$ as follows:

(i) $V' = V \bigcup_{a\in\Sigma} V_a$;

(ii) $T' = \bigcup_{a\in\Sigma} T_a$;

(iii) Production P' consists of the following:

(1) All productions in any P_a for each $a \in \Sigma$.

(2) All production of P but with each terminal "a'' in their bodies replaced by S_a wherever "a'' occurs.

Clearly, G' is a β-grammar of **order greater than** n. We claim that G' generates the language $f_s(L)$. The claim is proven if we show that a string $w \in L(G')$ iff $w \in f_s(L)$.

(If). Here we show that $f_s(L) \subseteq L(G')$. Suppose $w \in f_s(L)$. Then there is some string $x = a_1 a_2 \cdots a_n \in L$ and strings $x_i \in f_s(a_i)$ for $i = 1, 2, \cdots, n$ such that $w = x_a x_2 \cdots x_n$.

Now, corresponding to the string $a_1 a_2 \cdots a_n \in L = L(G)$, consider the sentential form $S_{a_1} S_{a_2} \cdots S_{a_n}$ derived in G' from the productions of G with S_a substituted for each a. Moreover, since the productions of each G_a are also productions of G', therefore, the derivation of x_i from S_{a_i} is also a derivation of G'. Hence $w = x_1 x_2 \cdots x_n$ is in $L(G')$.

(only-If). We show that $L(G') \subseteq f_s(L)$. Suppose $w \in L(G')$. Then we can identify a string $a_1 a_2 \cdots a_n$ in $L(G)$ and strings $x_i \in (f_s(a_i)$ such that $w = x_1 x_2 \cdots x_n$. But the string $x_1 x_2 \cdots x_n \in f_s(L)$ since it is formed by substituting string x_i for each of the a_i. Thus, we conclude that $w \in f_s(L)$ and therefore $L(G') \subseteq f_s(L)$.

Combining the two cases, we get

$$f_s(L) = L(G').$$

Hence $f_s(L) = L(G')$ is a β-language of **order greater than** n. Since the order of β-language L is n and order of β-language $f_s(L)$ is greater than n, therefore, the class of β-languages of order n is not closed under substitution operation. □

Example 3.2. Let $G = \beta(2) = (V, T, S, P)$ be a β-grammar of order 2 where

$$V = \{S, A\}, T = \{0, 1\},$$

and P is given by

$$S \to 0A; A \to 1A; A \to 1; A \to \lambda.$$

The β-language of order 2 generated by β-grammar $\beta(2)$ is given by

$$L = L(2) = \{01^n : n \geq 0\}.$$

Let f_s be the substitute function on $T = \{0, 1\}$ given by

$$f_s(0) = L_0 \text{ and } f_s(1) = L_1,$$

where L_0 and L_1 are both β-languages of order 2 given by

$$L_0(2)\{ab^k : k \geq 1\}$$

and

$$L_1(2) = \{cd^m : m \geq 1\}.$$

The β-language $L_0(2)$ is generated by the following β-grammar $G_0 = \beta_0(2)$ of order 2:

$$G_0 = \beta_0(2) = \{V_0, T_0, P_0, S_0\},$$

where

$$V_0 = \{Z, S_0\}, T_0 = \{a, b\},$$

and P_0 is given by

$$S_0 \to aZ;$$

$$Z \to bZ;$$

$$Z \to b.$$

Similarly, the β-langauge $L_1(2)$ is generated by the following β-grammar $G_1 = \beta_1(2)$ of order 2:

$$G_1 = \beta_1(2) = \{V_1, T_1, P_1, S_1),$$

where $V_1 = \{R, S_1\}, T = \{c, d\}$ and P_1 is given by

$$S_1 \to cR;$$

$$R \to dR;$$

$$R \to d.$$

Then the language $L' = f_s(L(2))$ obtained after applying the substitution operation f_s on β-language $L(2)$ is generated by the β-grammar $G'(3) = (V', T', P', S)$ of order 3, where

$$V' = \{S, S_0, S_1, R, Z, A\}, T' = \{0, 1, a, b, c, d\},$$

and P' is given by

$$\begin{aligned}
S_0 &\to aZ; \\
Z &\to bZ; \\
Z &\to b; \\
S_1 &\to cR; \\
R &\to dR; \\
R &\to d; \\
S &\to S_0A; \\
A &\to S_1A; \\
A &\to S_1; \\
A &\to \lambda.
\end{aligned}$$

The language $L'(3)$ is given by

$$L'(3) = \{ab^k(cd^m)^n : k, m \geq 1, n \geq 0\}.$$

Thus, the substitution operation does not preserve the order of the β-language $L(2)$.

4. Hyper β-language of order n

In the previous section, we have shown that the class of β-languages of order n is not closed under substitution operation since the order of the image β-language gets increased. In this section, we introduce the notion of hyper β-grammar and hyper β-language of order n and show that the class of hyper β-languages of order n is closed under substitution, union, concatenation and star-closure operations.

We define hyper β-grammar of order n as follows:

Definition 4.1. A **"hyper β-grammar of order n"** is a context-free grammar $G = (V, T, S, P)$ if all productions in P are of the form

$$A \to ax$$

where $a \in T \cup \{\lambda\}$ and $x \in V^*$ and any pair (A, a) for $a \in T$ occurs at most n times in P. A hyper β-grammar of order n will be denoted by $\beta_H(n)$.

Definition 4.2. The language generated by a hyper β-grammar of order n is called a **"hyper β-language of order n"**.

Remark 4.3. The only difference between a β-grammar and hyper β-grammar of order n is that in case of hyper β-grammar, there is no restriction on the number of productions of the form $A \to \lambda x$ or equivalently $A \to x$ for $x \in V^*$ and $A \in V$ whereas there can be at most n such productions in the case of β-grammar of order n.

Following result now directly follows from Definition 4.1 and Theorem 3.1:

Theorem 4.4. if L is a hyper β-language of order n over an alphabet Σ and f_s is a substitution function on Σ such that $f_s(a)$ is a hyper β-language of order n for each $a \in \Sigma$, then $f_s(L)$ is again a hyper β-language of order n , or equivalently, the class of hyper β-languages of order $n(n \geq 1)$ is closed under substitution operation. □

Remark 4.5. Under substitution operation, the order of hyper β-language is preserved while the order of β-language gets increased.

Theorem 4.6. *The family of hyper β-languages of order $n(n \geq 1)$ is closed under union, concatenation and star-closure.*

Proof. The proof is same as given in [5] for β-languages of order n. $\square$

5. Conclusion

In this paper, we have shown that the class of β-languages of order n is not closed under substitution operation. Motivated by this, we have introduced the notion of hyper β-language of order $n(n \geq 1)$. We have further shown that the class of hyper β-languages of order n is closed under substitution, union, concatenation and star-closure operations.

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